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ON UNIQUENESS OF SOLUTIONS FOR SECONDARY CREEP PROBLEMS†

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Abstract—Uniqueness results are established for solutions of secondary creep problems, including the effect of elastic strains, for a large class of domains subject to mixed boundary conditions. Two theorems are proved, one for quasistatic creep and one for dynamic.

1. INTRODUCTION

In [1], uniqueness was established for positive solutions of a certain nonlinear integral equation (eqn (1) of [1]) which governs the effective stress in internally loaded spherical and incompressible cylindrical pressure vessels subject to primary or secondary transient creep. It had been shown in [2, 3] that the solution of the latter boundary value problems can be reduced to this single equation.

In the present paper we prove uniqueness theorems for domains and loadings far more general than the above, however the case of primary creep has necessarily been excluded. It is assumed that the total infinitesimal strain tensor is the sum of an elastic strain and a creep strain, that the elastic strain-stress law is isotropic, and that the creep strain rate depends on the deviatoric components of the stress through a simple generalization of Norton's power law (eqn (2.2) below). The assumption of elastic isotropy has been made in order to be consistent with the isotropy of the creep response and can be relaxed considerably.

Details of the boundary value problems and hypotheses are given in Section 2. In Section 3, two theorems are proved. The first deals with the quasistatic case and establishes the uniqueness of the strain field. In the compressible case, stresses are also shown to be unique. For incompressible materials, we obtain uniqueness for the deviatoric components of the stress. From this, it follows that the stress field itself is unique up to a spatially constant hydrostatic pressure. Uniqueness of displacements follows provided that the part of the boundary on which displacements are prescribed in nontrivial.

The second theorem, included for the sake of completeness, furnishes uniqueness for the entire solution state: displacements, strains and stresses, for a dynamic creep problem in which the inertia term is included in the equations of motion and initial displacements and velocities are prescribed. Alternatively, one could prescribe displacement histories on $(-\infty, 0)$. The method of proof for both theorems is based on Gronwall's inequality[4] and was suggested by the work of Wheeler[5] on uniqueness of solutions for finite elastodynamics.

2. THE BOUNDARY VALUE PROBLEMS

Let R be a bounded region in three dimensional Euclidean space with closure $\bar{R}$ whose boundary ∂R is smooth enough to permit application of the Divergence Theorem. The field equations to be satisfied at all points $x \equiv (x_1, x_2, x_3)$ in R and times $t > 0$ are the strain-displacement relations‡

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.1}$$

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‡Subscripts have the range 1, 2, 3, repeated subscripts imply summation, δ_{ij} is the Kronecker delta, and for derivatives we write $\dot{v}_{i,j} \equiv (\partial^2 v_i)/(\partial t \partial x_j)$.

the strain-stress relations

$$\epsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}] + \int_0^t F(\sigma_e)s_{ij}\,d\tau, \tag{2.2}$$

and *either* the quasistatic equations of motion

$$\sigma_{ij,j} + b_i = 0, \quad \sigma_{ij} = \sigma_{ji}, \tag{2.3Q}$$

or the dynamic equations of motion

$$\sigma_{ij,j} + b_i = \rho\ddot{u}_i, \quad \sigma_{ij} = \sigma_{ji}. \tag{2.3D}$$

Here u_i, ϵ_{ij}, σ_{ij} denote, respectively, the components of the displacement vector, the infinitesimal strain tensor, and the stress tensor. The constants ν, E are Poisson's ratio and Young's modulus respectively, and are subject to the restrictions

$$-1 < \nu \leq \frac{1}{2}, \quad E > 0. \tag{2.4}$$

F is a creep response function assumed to be continuously differentiable on $[0, \infty)$. Its argument, σ_e, is called the "effective stress" and is defined by

$$\sigma_e^2 = \frac{3}{2}s_{ij}s_{ij} \tag{2.5}$$

where s_{ij}, the deviatoric components of the stress, are given by

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk}. \tag{2.6}$$

In the special case in which the creep response is governed by the well-known Norton power law[6] with exponent n, F becomes

$$F(\sigma_e) = \frac{3}{2}k\sigma_e^{n-1} \quad (n \geq 2,\ k > 0).$$

The law (2.2) is interpreted as having the form

$$\epsilon_{ij} = \epsilon_{ij}^{(e)} + \epsilon_{ij}^{(c)} \tag{2.7}$$

where $\epsilon_{ij}^{(e)}$ denotes the "elastic" strains and $\epsilon_{ij}^{(c)}$ the "creep" strains. In (2.3), b_i stands for the body force per unit volume and ρ for the mass density. It is assumed that $\rho(x) > 0$ on $\bar{R}$.

For the *quasistatic* problem, the field equations (2.1), (2.2), (2.3Q) are assumed to hold in R not only for $t > 0$ but also for $t = 0$. For the *dynamic* problem (2.1), (2.2), (2.3D), we have instead at $t = 0$ the initial conditions

$$u_i(x, 0) = \phi_i(x), \quad \dot{u}_i(x, 0) = \psi_i(x) \tag{2.8}$$

for all x in $\bar{R}$. To either the quasistatic or the dynamic problem we adjoin boundary conditions

$$u_i(x, t) = f_i(x, t) \quad (x \text{ in } \partial R_1, t > 0) \tag{2.9}$$

$$\sigma_{ij}n_j(x, t) = g_i(x, t) \quad (x \text{ in } \partial R_2, t > 0) \tag{2.10}$$

where ∂R_1 and ∂R_2 are mutually exclusive subsets of ∂R whose union equals ∂R. In (2.10), $n_i(x)$ are the components of the unit outward normal vector at point x on ∂R.

3. THE UNIQUENESS THEOREMS

In order to eliminate needless repetition in the proofs of the uniqueness theorems, we shall first establish the following inequality.

Lemma 1. *Let ψ_{ij}, $\hat{\sigma}_{ij}$, and $\tilde{\sigma}_{ij}$ be continuous tensor fields on $\bar{R}\times[0,\infty)$ and let*

$$\bar{\sigma}_{ij}\equiv\hat{\sigma}_{ij}-\tilde{\sigma}_{ij},\quad \bar{s}_{ij}\equiv\hat{s}_{ij}-\tilde{s}_{ij} \tag{3.1}$$

where

$$\hat{s}_{ij}=\hat{\sigma}_{ij}-\frac{\delta_{ij}}{3}\hat{\sigma}_{kk},\quad \hat{\sigma}_e^{\,2}=\frac{3}{2}\hat{s}_{ij}\hat{s}_{ij},\ \text{etc.}$$

Then, given $T>0$, there exists a finite constant $C(T)$ (which may depend on $\hat{\sigma}_{ij}$, $\tilde{\sigma}_{ij}$) such that for any $0\le\tau\le T$ and $t\ge0$,

$$\int_R|\psi_{ij}(x,t)||F(\hat{\sigma}_e)\hat{s}_{ij}-F(\tilde{\sigma}_e)\tilde{s}_{ij}|(x,\tau)\,\mathrm{d}x\le C(T)\left[\int_R\psi_{ij}\psi_{ij}(x,t)\,\mathrm{d}x\right]^{1/2}\left[\int_R\bar{s}_{ij}\bar{s}_{ij}(x,\tau)\,\mathrm{d}x\right]^{1/2}. \tag{3.2}$$

Proof. By the Mean Value Theorem,

$$\begin{aligned}|F(\hat{\sigma}_e)\hat{s}_{ij}-F(\tilde{\sigma}_e)\tilde{s}_{ij}|&=|F'(\zeta)[\hat{\sigma}_e-\tilde{\sigma}_e]\hat{s}_{ij}+F(\tilde{\sigma}_e)\bar{s}_{ij}|\\&\le M(T)|\hat{s}_{ij}|\bar{\sigma}_e+|F(\tilde{\sigma}_e)\bar{s}_{ij}|.\end{aligned} \tag{3.3}$$

Here $\zeta(x,t)$ is some intermediate value between $\hat{\sigma}_e(x,t)$ and $\tilde{\sigma}_e(x,t)$ and $M(T)$ is a finite constant. For second order tensor fields $v_{ij}(x)$, $w_{ij}(x)$ on R, consider the inner product

$$(\underline{v},\underline{w})\equiv\int_R v_{ij}(x)w_{ij}(x)\,\mathrm{d}x.$$

Using the Schwarz and Minkowski inequalities corresponding to this inner product together with (3.3), we obtain

$$\begin{aligned}\int_R|\psi_{ij}(x,t)||F(\hat{\sigma}_e)\hat{s}_{ij}-F(\tilde{\sigma}_e)\tilde{s}_{ij}|(x,\tau)\,\mathrm{d}x\le&\left[\int_R\psi_{ij}\psi_{ij}(x,t)\,\mathrm{d}x\right]^{1/2}\left\{\left[\int_R(M(T)\hat{\sigma}_e)^2\bar{s}_{ij}\bar{s}_{ij}(x,\tau)\,\mathrm{d}x\right]^{1/2}\right.\\&\left.+\left[\int_R F^2(\tilde{\sigma}_e)\bar{s}_{ij}\bar{s}_{ij}(x,\tau)\,\mathrm{d}x\right]^{1/2}\right\}.\end{aligned}$$

Here we have used the fact that

$$\bar{\sigma}_e^{\,2}\hat{s}_{ij}\hat{s}_{ij}=\hat{\sigma}_e^{\,2}\bar{s}_{ij}\bar{s}_{ij}.$$

Due to the assumed smoothness of $\tilde{\sigma}_{ij}$, $\hat{\sigma}_{ij}$ and F, (3.2) follows immediately from the above inequality.

Theorem 1. *Let $S\equiv[u_i,\epsilon_{ij},\sigma_{ij}]$ be a continuously differentiable solution of the quasistatic problem consisting of eqns* (2.1), (2.2) *and* (2.3Q) *together with the boundary conditions* (2.9), (2.10). *Then the strain field ϵ_{ij} is uniquely determined. Furthermore, for $\nu<0.5$, the stresses σ_{ij} are unique. For $\nu=0.5$, the deviatoric components s_{ij} are unique.*

Proof. Let $\tilde{S}\equiv[\tilde{u}_i,\tilde{\epsilon}_{ij},\tilde{\sigma}_{ij}]$ and $\hat{S}\equiv[\hat{u}_i,\hat{\epsilon}_{ij},\hat{\sigma}_{ij}]$ be two continuously differentiable solutions of the boundary value problem in question and define the difference state $\bar{S}$ as in (3.1). If (2.2) is applied to the $\hat{S}$ quantities and then to $\tilde{S}$, the difference between the resulting two equations is

$$\bar{\epsilon}_{ij}=\frac{1}{E}[(1+\nu)\bar{\sigma}_{ij}-\nu\delta_{ij}\bar{\sigma}_{kk}]+\int_0^t[F(\hat{\sigma}_e)\hat{s}_{ij}-F(\tilde{\sigma}_e)\tilde{s}_{ij}]\,\mathrm{d}\tau. \tag{3.4}$$

In the incompressible case $\nu = 1/2$, (3.4) takes the form

$$\bar{\epsilon}_{ij} = \frac{3}{2E}\bar{s}_{ij} + \int_0^t [F(\hat{\sigma}_e)\hat{s}_{ij} - F(\tilde{\sigma}_e)\tilde{s}_{ij}]\,d\tau. \tag{3.4I}$$

We now multiply both sides of (3.4) by $\bar{\sigma}_{ij}(x, t)$ and integrate over R. Since $\bar{u}_i = 0$ on ∂R_1 and $\bar{\sigma}_{ij}n_j = 0$ on ∂R_2 for all $t > 0$, we can use (2.1), (2.3Q) and the Divergence Theorem to obtain

$$0 = \frac{1}{E}\int_R [(1+\nu)\bar{\sigma}_{ij}\bar{\sigma}_{ij} - \nu(\bar{\sigma}_{kk})^2](x, t)\,dx + \int_R\int_0^t \bar{\sigma}_{ij}(x, t)[F(\hat{\sigma}_e)\hat{s}_{ij} - F(\tilde{\sigma}_e)\tilde{s}_{ij}](x, \tau)\,d\tau\,dx. \tag{3.5}$$

The same reasoning applied to (3.4I) yields the identity

$$0 = \frac{3}{2E}\int_R \bar{s}_{ij}\bar{s}_{ij}(x, t)\,dx + \int_R\int_0^t \bar{s}_{ij}(x, t)[F(\hat{\sigma}_e)\hat{s}_{ij} - F(\tilde{\sigma}_e)\tilde{s}_{ij}](x, \tau)\,d\tau\,dx. \tag{3.5I}$$

Taking first the case $\nu < 1/2$, we define

$$v^2(t) = \int_R \bar{\sigma}_{ij}\bar{\sigma}_{ij}(x, t)\,dx. \tag{3.6}$$

By (3.4) the proof for this case will be complete, once we show that $v \equiv 0$ on $[0, T]$ for any finite $T > 0$. For this, it suffices, due to Gronwall's lemma, to show that v satisfies an inequality of the form

$$v(t) \le M\int_0^t v(\tau)\,d\tau \quad (0 \le t \le T) \tag{3.7}$$

for some finite constant $M = M(T)$.

To establish (3.7) for $T > 0$ given, we apply Lemma 1 to the second integral in (3.5) with the choice $\psi_{ij} \equiv \bar{\sigma}_{ij}$ to obtain

$$\frac{1}{E}\int_R [(1+\nu)\bar{\sigma}_{ij}\bar{\sigma}_{ij} - \nu(\bar{\sigma}_{kk})^2](x, t)\,dx \le C(T)v(t)\int_0^t \left[\int_R \bar{s}_{ij}\bar{s}_{ij}(x, \tau)\,dx\right]^{1/2} d\tau \tag{3.8}$$

for $0 \le t \le T$. Since

$$(\bar{\sigma}_{kk})^2 \le 3\bar{\sigma}_{ij}\bar{\sigma}_{ij}, \quad \bar{s}_{ij}\bar{s}_{ij} = \bar{\sigma}_{ij}\bar{\sigma}_{ij} - \frac{(\bar{\sigma}_{kk})^2}{3}, \tag{3.9}$$

it follows from (3.8) that

$$\frac{(1-2\nu)}{E}v^2(t) \le C(T)v(t)\int_0^t v(\tau)\,d\tau,$$

from which (3.7) is immediate.

In the case $\nu = 1/2$, we apply Lemma 1 to the second integral in (3.5I) with $\psi_{ij} \equiv \bar{s}_{ij}$ to show that the quantity

$$w^2(t) \equiv \int_R \bar{s}_{ij}\bar{s}_{ij}(x, t)\,dx$$

satisfies a Gronwall-type inequality. From (3.4I) it is clear that $w \equiv 0$ implies $\bar{\epsilon}_{ij} \equiv 0$. This completes the proof.

Theorem 2. *Let* $S \equiv [u_i, \epsilon_{ij}, \sigma_{ij}]$ *be a solution of the dynamic problem* (2.1), (2.2) *and* (2.3D) *which is* C^2 *in* $\bar{R} \times [0, \infty)$ *and satisfies the boundary conditions* (2.9), (2.10) *on* $\partial R \times (0, \infty)$ *and the initial conditions* (2.8) *in* $\bar{R}$. *Then S is uniquely determined.*

Proof. We shall discuss only the case $\nu < 1/2$. We again suppose the existence of two distinct solutions, $\tilde{S}$ and $\hat{S}$. With the notation of Theorem 1, (3.4) again holds. For the dynamic case, we differentiate (3.4) with respect to t, multiply both sides of the resulting equation by $\bar{\sigma}_{ij}$ and integrate over R. Thus,

$$\int_R \dot{\bar{\epsilon}}_{ij}\bar{\sigma}_{ij}\,\mathrm{d}x = \frac{1}{E}\int_R [(1+\nu)\bar{\sigma}_{ij}\dot{\bar{\sigma}}_{ij} - \nu\bar{\sigma}_{kk}\dot{\bar{\sigma}}_{ll}]\,\mathrm{d}x + \int_R \bar{\sigma}_{ij}[F(\hat{\sigma}_e)\hat{s}_{ij} - F(\tilde{\sigma}_e)\tilde{s}_{ij}]\,\mathrm{d}x. \tag{3.10}$$

Since the difference state $\bar{S}$ satisfies homogeneous boundary conditions and the homogeneous equations of motion

$$\bar{\sigma}_{ij,j} = \rho\ddot{\bar{u}}_i,$$

we can apply the Divergence Theorem in the usual way to the integral on the left-hand side of (3.10) to establish that

$$\int_R \dot{\bar{\epsilon}}_{ij}\bar{\sigma}_{ij}\,\mathrm{d}x = -\int_R \rho\dot{\bar{u}}_i\ddot{\bar{u}}_i\,\mathrm{d}x.$$

This fact enables us to put (3.10) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_R \frac{\rho}{2}\dot{\bar{u}}_i\dot{\bar{u}}_i\,\mathrm{d}x + \frac{1}{2E}\int_R [(1+\nu)\bar{\sigma}_{ij}\bar{\sigma}_{ij} - \nu(\bar{\sigma}_{kk})^2]\,\mathrm{d}x\right\} = -\int_R \bar{\sigma}_{ij}[F(\hat{\sigma}_e)\hat{s}_{ij} - F(\tilde{\sigma}_e)\tilde{s}_{ij}]\,\mathrm{d}x. \tag{3.11}$$

Since (2.8) implies that

$$\bar{u}_i(x, 0) = \dot{\bar{u}}_i(x, 0) = 0 \tag{3.12}$$

in $\bar{R}$, it follows from (2.1) and the smoothness assumptions that

$$\bar{\epsilon}_{ij}(x, 0+) = 0$$

in $\bar{R}$. Therefore, letting $t \to 0$ in (3.4) and using the constitutive assumptions $E > 0$, $-1 < \nu < 1/2$, we see that

$$\bar{\sigma}_{ij}(x, 0+) = 0.$$

Therefore, if we integrate (3.11) from 0 to t, we obtain

$$\int_R \frac{\rho}{2}\dot{\bar{u}}_i\dot{\bar{u}}_i(x,t)\,\mathrm{d}x + \frac{1}{2E}\int_R [(1+\nu)\bar{\sigma}_{ij}\bar{\sigma}_{ij} - \nu(\bar{\sigma}_{kk})^2](x,t)\,\mathrm{d}x$$
$$= -\int_0^t \int_R \bar{\sigma}_{ij}[F(\hat{\sigma}_e)\hat{s}_{ij} - F(\tilde{\sigma}_e)\tilde{s}_{ij}](x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{3.13}$$

We now apply Lemma 1 to the right-hand side of (3.13) with $\psi_{ij} \equiv \bar{\sigma}_{ij}$ and $t = \tau \le T$ for some prescribed T. It follows that

$$\int_R \frac{\rho}{2}\dot{\bar{u}}_i\dot{\bar{u}}_i(x,t)\,\mathrm{d}x + \frac{1}{2E}\int_R [(1+\nu)\bar{\sigma}_{ij}\bar{\sigma}_{ij} - \nu(\bar{\sigma}_{kk})^2](x,t)\,\mathrm{d}x$$
$$\le C(T)\int_0^t \left[\int_R \bar{\sigma}_{ij}\bar{\sigma}_{ij}(x,\tau)\,\mathrm{d}x\right]^{1/2}\left[\int_R \bar{s}_{ij}\bar{s}_{ij}(x,\tau)\,\mathrm{d}x\right]^{1/2}\mathrm{d}\tau. \tag{3.14}$$

Define

$$z(t)=\int_R \frac{\rho}{2}\dot{\bar{u}}_i\dot{\bar{u}}_i(x,t)\,\mathrm{d}x+\int_R \bar{\sigma}_{ij}\bar{\sigma}_{ij}(x,t)\,\mathrm{d}x.$$

The theorem will follow, provided we show that $z\equiv 0$. However it is easy to fashion a Gronwall inequality for $z(t)$ from (3.14) using (3.9) and the assumption $\nu<1/2$. This completes the proof.

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